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Relativity and spin–orbit interaction in nuclei†

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Abstract. It is shown that the observed spin–orbit splittings in the atomic nuclei may be explained theoretically if one takes into account the relativistic character of the nucleon motion in the bound many-nucleon system. The relativistic corrections were obtained from the approximately relativistically invariant (to within terms in v^2/c^2) Breit-type equation for two nucleons. Two-particle interactions were taken in the form of one-boson-exchange potentials which reproduce well the nucleon–nucleon scattering data and which include pseudoscalar, scalar and vector mesons and resonances. The calculations of the doublet splittings of different single-particle levels in ^{41}Ca and ^{209}Pb were carried out, the values of the nuclear average field being calculated as well. Agreement with experiment is reasonably satisfactory.

1. Introduction

In all modern 'microscopic' nuclear models the states of a single-particle potential are used as a basis, the operator of the single-particle spin–orbit interaction being an essential part of this potential. However, although many years have passed since the spin–orbit interaction, which appeared to be the starting point for the explanation of the 'magic numbers' and shell model creation, was introduced in nuclear physics by Goeppert-Mayer (1949, 1950) and Haxel *et al* (1949, 1950), up to the present the origin of the spin–orbit interaction in the nucleus is not quite clear.

At first it was natural to interpret this interaction by analogy with that of the atom, where the greater part of the observed spin–orbit coupling is treated as a relativistic correction to electron motion in a self-consistent average field (the Thomas coupling). Furry (1936) explained the opposite signs of the doublet splittings in atomic and nuclear spectra as a result of different types of interaction of particles: the electrostatic potential experienced by an electron in an atom transforms as the fourth component of the vector electromagnetic field; whereas, following Furry, we may regard the nuclear potential to be a scalar. However, these early attempts to ascribe a relativistic origin to the spin–orbit interaction in the nucleus were given up when the calculations showed that the Thomas coupling gives doublet splittings that are 30–40 times smaller than those actually observed in the nuclei (Fernbach 1958).

In the present paper the authors return to the relativistic treatment of the single-particle spin–orbit interaction in the nucleus but on a basis different from that used by

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previous authors. The principal points of our treatment are: (i) the derivation of the *relativistic corrections* (of the order v^2/c^2) to the relative motion of two nucleons coupled in the nucleus by the *two-particle conservative forces*, (ii) inclusion of the corrections obtained into the conventional pattern of the *many-nucleon problem*. Further, we derive the single-particle spin–orbit operator (in this point the procedure is similar to that applied in the phenomenological approach to the problem (Blin-Stoyle 1955)) and using this operator we calculate the spin–orbit splittings in the near magic nuclei. Furthermore we calculate the average nuclear field (in the Hartree approximation) on the basis of two-particle forces. We have utilized two-particle forces of different types. We have started from the forces of the conventional type which are commonly used in the shell model calculations etc. These forces were found to give too small spin–orbit splittings, though leading to reasonable values of the average field. Then we used as two-particle forces one-boson-exchange potentials with different mesons and resonances taken into account. Lately these potentials have been successfully applied (see eg Green and Sawada 1967, Ueda and Green 1968, Ingber 1968, Bryan and Scott 1969, Ingber and Potenza 1970) to the description of nucleon–nucleon scattering within a wide scattering region (0–400 MeV incident lab energy), deuteron data etc with a moderate number of adjustable parameters. It was found by us (see § 4) that such potentials lead to rather reasonable values of the spin–orbit interaction and at the same time give the values of the average field close to the experimental values.

2. Relativistic corrections in the many-nucleon problem and spin–orbit potential

It is obvious from the following simplified estimate that the nucleons in the nuclei may move with high (semirelativistic) velocities: if the depth of the potential well is assumed to be equal to 50–60 MeV, the kinetic energy of the 'highest' nucleons being equal to 40–50 MeV, then we have $v \simeq \frac{1}{3}c$.

If one considers the ith nucleon as moving in the field formed by the two-particle interactions V_{ik} with the other nucleons, that is, in the field $V(i) = \Sigma_{k(k\neq i)} V_{ik}$, the relativistic equation describing such motion is the Dirac equation for a particle in an external field $V(i)$. Regarding the nuclear field $V(i)$ as a scalar, the relativistic equation for a nucleon moving in the field $V(i)$ is:

$$i\hbar\frac{\partial\phi}{\partial t} = \left\{c\boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta\left(mc^2 + \sum_{k(k\neq i)} V_{ik}\right)\right\}\phi, \tag{1}$$

where matrices $\boldsymbol{\alpha}$ and β have a conventional form (Achieser and Berestetsky 1959), and ϕ is a four-component function composed of two spinors: 'large' component ψ and 'small' component χ (Achieser and Berestetsky 1959).

However, this manner of consideration is based on the assumption which is equivalent to a 'freezing' of the relativistic motion of all the nucleons with the exception of the 'test' i nucleon. It is obvious that such an assumption would not be proper. To obtain the relativistic corrections in a consistent manner, we shall consider the approximately relativistic equation (relativistically invariant to within terms in v^2/c^2) for two nucleons interacting through the scalar potential V_{ik} (we shall start from the scalar interaction since it is easier to explain the main features of our description using this example). This equation was derived by Breit (1937) by analogy with the approximate relativistic

Breit equation for two electrons and has the following form:

$$\left(c(\boldsymbol{\alpha}_i \cdot \boldsymbol{p}_i + \boldsymbol{\alpha}_k \cdot \boldsymbol{p}_k) + (\beta_i + \beta_k)mc^2 - \beta_i\beta_k V_{ik} - \frac{1}{2}\boldsymbol{\alpha}_i \cdot \boldsymbol{\alpha}_k V_{ik} - \frac{1}{2}(\boldsymbol{\alpha}_i \cdot \boldsymbol{r}_{ik})(\boldsymbol{\alpha}_k \cdot \boldsymbol{r}_{ik})\frac{1}{r_{ik}}\frac{\mathrm{d}V_{ik}}{\mathrm{d}r_{ik}}\right)\phi_{ik} = E\phi_{ik} \tag{2}$$

where $\boldsymbol{r}_{ik} = \boldsymbol{r}_i - \boldsymbol{r}_k$, $-E = \epsilon + 2mc^2$ the total energy of the system.

The wave equation (2) can be reduced to an equation for only the 'large' component ψ correct to the terms of the order v^2/c^2 (Achieser and Berestetsky 1959, Breit 1937). In this case we obtain

$$\left(\frac{\boldsymbol{p}_i^2 + \boldsymbol{p}_k^2}{2m} + V_{ik} + V_{ik}^{\mathrm{so}} + \delta V_{ik}^{\mathrm{rel}}\right)\psi = \epsilon\psi, \tag{3a}$$

where

$$V_{ik}^{\mathrm{so}} = -\frac{\hbar}{4m^2c^2}\frac{1}{r_{ik}}\frac{\mathrm{d}V_{ik}}{\mathrm{d}r_{ik}}\{(\boldsymbol{r}_i - \boldsymbol{r}_k) \times \boldsymbol{p}_i \cdot \boldsymbol{\sigma}_i - (\boldsymbol{r}_i - \boldsymbol{r}_k) \times \boldsymbol{p}_k \cdot \boldsymbol{\sigma}_k\}, \tag{3b}$$

$\delta V_{ik}^{\mathrm{rel}}$ contains all relativistic corrections which do not possess the form of V_{ik}^{so}; $\boldsymbol{\sigma}$ are the Pauli matrices.

In the centre of mass system $\boldsymbol{p}_i + \boldsymbol{p}_k = 0$. Then taking into account the following identity:

$$\boldsymbol{\sigma}_{\substack{i \\ k}} = \tfrac{1}{2}(\boldsymbol{\sigma}_i + \boldsymbol{\sigma}_k) + \tfrac{1}{2}(\boldsymbol{\sigma}_{\substack{i \\ k}} - \boldsymbol{\sigma}_{\substack{k \\ i}}), \tag{4}$$

in the case of the central forces we shall obtain from equation (3*b*):

$$V_{ik}^{\mathrm{so}} \to -\frac{\hbar^2}{4m^2c^2}\frac{1}{r_{ik}}\frac{\mathrm{d}V_{ik}}{\mathrm{d}r_{ik}}(\boldsymbol{\sigma}_i + \boldsymbol{\sigma}_k)\boldsymbol{l}_{ik}, \tag{5}$$

where $\boldsymbol{l}_{ik} = (\boldsymbol{r}_{ik} \times \boldsymbol{p}_{ik})/2\hbar$ is the operator of the relative orbital motion of two nucleons. However, considering a many-particle system we must use equation (3*b*) rather than its particular case (5).

From equation (3*a*) for the many-nucleon system we have:

$$H = \sum_i \frac{\boldsymbol{p}_i^2}{2m} + \sum_{i<k} V_{ik} + \sum_{i<k} \delta V_{ik}^{\mathrm{rel}} + \sum_{i<k} V_{ik}^{\mathrm{so}}. \tag{6}$$

Replacing, in accordance with the conventional approach, $\Sigma_{k(k\neq i)} V_{ik}$ by the average field and residual interactions

$$\sum_{k(k\neq i)} V_{ik} = V_{\mathrm{av}}(i) + V_{\mathrm{res}}(i) \tag{7}$$

and formally including the operator $\Sigma_{k(k\neq i)} \delta V_{ik}^{\mathrm{rel}}$ into the residual interactions, we get the following Hamiltonian for the ith nucleon:

$$H(i) = \frac{\boldsymbol{p}_i^2}{2m} + V_{\mathrm{av}}(i) + V_{\mathrm{res}}(i) + \sum_{k(k\neq i)} V_{ik}^{\mathrm{so}}. \tag{8}$$

In nuclei the *jj* coupling is realized, that is, the spin–orbit interaction is stronger than the residual interaction. For this reason it is possible, in the first approximation, to omit $V_{\mathrm{res}}(i)$ and to consider the expectation value of the spin–orbit interaction of the ith nucleon

with the remainder of the nucleons on the basis of the functions $\Psi^{(0)} = \Pi_{k(k\neq i)}\psi_k^{(0)}$, where $\psi_k^{(0)}$ is the solution of the wave equation for the average field $V_{\text{av}}(k)$:

$$H_k^{(0)}\psi_k^{(0)} = \epsilon_k^{(0)}\psi_k^{(0)}$$

$$H_k^{(0)} = \frac{\boldsymbol{p}_k^2}{2m} + V_{\text{av}}(k) \tag{9}$$

where the nucleon suffix denotes simultaneously the set of quantum numbers for a single particle. This expectation value gives the single-particle operator of the spin–orbit interaction:

$$V_{\text{so}}^{(0)}(i) = \left\langle \Psi^{(0)} \middle| \sum_{k(k\neq i)} V_{ik}^{\text{so}} \middle| \Psi^{(0)} \right\rangle. \tag{10}$$

Further, in accordance with the usual self-consistency procedure one should take the eigenfunctions $\psi_k^{(1)}$ of the Hamiltonian

$$H_k^{(1)} = \frac{\boldsymbol{p}_k^2}{2m} + V_{\text{av}}(k) + V_{\text{so}}^{(0)}(k) \tag{11}$$

as the basis functions and obtain:

$$V_{\text{so}}^{(1)}(i) = \left\langle \Psi^{(1)} \middle| \sum_{k(k\neq i)} V_{ik}^{\text{so}} \middle| \Psi^{(1)} \right\rangle \tag{12}$$

where $\Psi^{(1)} = \Pi_{k(k\neq i)}\psi_k^{(1)}$, and so forth.

Furthermore, in the self-consistency procedure following $V_{\text{so}}^{(0)}(i)$ one should take into consideration both the correct symmetry of the states and the residual interactions, that is, performing the replacement

$$\sum_i \left(\sum_{k>i} V_{ik} - V_{\text{av}}(i) \right) = \tfrac{1}{2} \sum_{i,k} V_{ik}^{\text{res}} \tag{13a}$$

$$\sum_i \left(\sum_{k>i} V_{ik}^{\text{so}} - V_{\text{so}}^{(0)}(i) \right) = \tfrac{1}{2} \sum_{i,k} \overset{\text{so}}{V}{}_{ik}^{\text{res}}, \tag{13b}$$

one should solve the wave equation for the complete Hamiltonian (we give it in the representation of the secondary quantization):

$$H = \sum_i \epsilon_i^{(1)} a_i^+ a_i + \tfrac{1}{2} \sum_{iklm} \langle ik| V_{\xi\xi'}^{\text{res}} + \overset{\text{so}}{V}{}_{\xi\xi'}^{\text{res}} |lm\rangle a_i^+ a_k^+ a_l a_m, \tag{14}$$

$\epsilon_i^{(1)}$ and $|i\rangle$ being, respectively, the eigenvalues and eigenfunctions of the Hamiltonian $H_i^{(1)}$; $a_i^+(a_i)$ are the operators of creation (annihilation) of a particle in the state i. However, such a problem would be very cumbersome. For this reason, in the present paper we shall limit ourselves to the calculation of $V_{\text{so}}^{(0)}(i)$ for the near magic nuclei, leaving the realization of the above mentioned program for the future.

Let us consider the wavefunction $\Psi^{(0)} = \Pi_{k(k\neq i)}\psi_k^{(0)}$ of the spin saturated core which is formed by $A-1$ nucleons, the ith nucleon moving outside the core. In this case the expectation values of $\Sigma_{k(k\neq i)}\boldsymbol{\sigma}_k$ and $\Sigma_{k(k\neq i)}\boldsymbol{p}_k$ will be equal to zero, and we obtain

$$V_{\text{so}}^{(0)}(i) = \frac{\hbar}{4m^2c^2}\boldsymbol{\sigma}_i \cdot \boldsymbol{r}_i \times \boldsymbol{p}_i \int \rho^{(0)}(\boldsymbol{r}) \frac{1}{x}\frac{\mathrm{d}V_{ik}}{\mathrm{d}x}\left(\frac{\boldsymbol{r}\cdot\boldsymbol{r}_i}{r_i^2} - 1\right) \mathrm{d}\boldsymbol{r}, \tag{15}$$

where

$$\boldsymbol{x} = \boldsymbol{r} - \boldsymbol{r}_i$$

$$\rho^{(0)}(\boldsymbol{r}) = \sum_{k(k\neq i)} |\psi_k^{(0)}(\boldsymbol{r})|^2.$$

Expressing $\rho^{(0)}(\boldsymbol{r})$ in powers of $\boldsymbol{x}$, limiting the expansion to the first three terms

$$\rho^{(0)}(\boldsymbol{r}) = \rho^{(0)}(\boldsymbol{r}_i) + (\nabla\rho^{(0)})_{\boldsymbol{r}=\boldsymbol{r}_i} \cdot \boldsymbol{x} + \tfrac{1}{2} \sum_{\mu,\nu=1}^{3} \left(\frac{\partial^2 \rho^{(0)}}{\partial x_\mu \, \partial x_\nu}\right)_{\boldsymbol{r}=\boldsymbol{r}_i} x_\mu x_\nu \tag{16}$$

and taking into consideration the fact that because of the parity of the integrand only the term with $\nabla\rho^{(0)}$ contributes, we have

$$V_{\mathrm{so}}^{(0)}(i) \equiv V_{\mathrm{so}}^{(0)}(\boldsymbol{r}) = \frac{\pi\hbar^2}{3m^2c^2} \frac{1}{r} \frac{\mathrm{d}\rho^{(0)}}{\mathrm{d}r} \left(\int_0^\infty \frac{\mathrm{d}V_{ik}}{\mathrm{d}x} x^3 \, \mathrm{d}x \right) \boldsymbol{l} \cdot \boldsymbol{\sigma}. \tag{17}$$

Up to the present we have considered the case when V_{ik} is a scalar interaction. We shall see in what follows (see §§ 3 and 4) that a scalar interaction is insufficient for the simultaneous derivation of reasonable values of the spin–orbit interaction and the average field. That is why we shall generalize the consideration given above for the case of the two-particle one-boson-exchange interaction which includes the exchange of pseudoscalar (P), vector (V) and scalar (S) mesons and resonances. As was already mentioned in § 1, one-boson-exchange potentials (OBEP) with P, V, S mesons taken into account have been widely used in recent years. In accordance with Green and Sawada (1967) for two nucleons interacting via OBEP instead of equation (2) we have

$$\begin{aligned} \Big\{ & c(\boldsymbol{\alpha}_i \cdot \boldsymbol{p}_i + \boldsymbol{\alpha}_k \cdot \boldsymbol{p}_k) + (\beta_i + \beta_k)mc^2 - \beta_i\beta_k \left(\sum_{\mathrm{S}} V_{ik}^{\mathrm{S}} + \boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_k \sum_{(\mathrm{S}\tau)} V_{ik}^{\mathrm{S}\tau} \right) \\ & + (1 - \boldsymbol{\alpha}_i \cdot \boldsymbol{\alpha}_k) \left(\sum_{\mathrm{V}} V_{ik}^{\mathrm{V}} + \boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_k \sum_{(\mathrm{V}\tau)} V_{ik}^{\mathrm{V}\tau} \right) \\ & + \beta_i\gamma_{5,i}\beta_k\gamma_{5,k} \left(\sum_{\mathrm{P}} V_{ik}^{\mathrm{P}} + \boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_k \sum_{(\mathrm{P}\tau)} V_{ik}^{\mathrm{P}\tau} \right) \Big\} \phi_{ik} = E\phi_{ik}, \end{aligned} \tag{18}$$

where $\gamma_5 = \beta\gamma^1\gamma^2\gamma^3$, $\gamma^l (l = 1, 2, 3)$ being the components of the matrix $\boldsymbol{\gamma} = \beta\boldsymbol{\alpha}$; $\boldsymbol{\tau}_i$ is the isospin matrix of the ith nucleon.

The interactions written out in equation (18) correspond to the exchange of pseudoscalar (V_{ik}^{P}, $V_{ik}^{\mathrm{P}\tau}$), vector (V_{ik}^{V}, $V_{ik}^{\mathrm{V}\tau}$) and scalar (V_{ik}^{S}, $V_{ik}^{\mathrm{S}\tau}$) mesons, which may be both isoscalars and isovectors. The concrete form of the functions V_{ik} is considered in § 4.

Now we shall reduce (to within terms in v^2/c^2) equation (18) to the equation for the 'large' component ψ only (similarly to the reduction of equation (2) to equation (3*a*)). It is to be emphasized that in the case of vector mesons the operator of the Larmor type

$$\frac{\hbar}{2m^2c^2} \frac{1}{r_{ik}} \frac{\mathrm{d}V_{ik}^{\mathrm{V}}}{\mathrm{d}r_{ik}} (\boldsymbol{r}_{ik} \times \boldsymbol{p}_k \cdot \boldsymbol{\sigma}_i - \boldsymbol{r}_{ik} \times \boldsymbol{p}_i \cdot \boldsymbol{\sigma}_k) \tag{19}$$

should be added to the operator of the spin–orbit interaction of the Thomas type

$$-\frac{\hbar}{4m^2c^2} \frac{1}{r_{ik}} \frac{\mathrm{d}V_{ik}^{\mathrm{V}}}{\mathrm{d}r_{ik}} (\boldsymbol{r}_{ik} \times \boldsymbol{p}_i \cdot \boldsymbol{\sigma}_i - \boldsymbol{r}_{ik} \times \boldsymbol{p}_k \cdot \boldsymbol{\sigma}_k). \tag{20}$$

However, in the Hartree approximation the operator of the Larmor type makes no contribution to the single-particle spin–orbit potential for the nuclei having one nucleon

above the closed shells (in this paper we shall limit ourselves to the consideration of such nuclei only). Taking into account all the aforesaid we shall obtain instead of equation (3*a*) the following equation (the details of transformations of the operators may be found, for example, in the survey by Green and Sawada (1967)):

$$\left[\frac{\boldsymbol{p}_i^2+\boldsymbol{p}_k^2}{2m}+\left\{\left(\sum_{\mathrm{S}} V_{ik}^{\mathrm{S}}-\sum_{\mathrm{V}} V_{ik}^{\mathrm{V}}\right)+\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_k\left(\sum_{(\mathrm{S}\tau)} V_{ik}^{\mathrm{S}\tau}-\sum_{(\mathrm{V}\tau)} V_{ik}^{\mathrm{V}\tau}\right)\right\}\right.$$
$$-\frac{\hbar}{4m^2c^2}(\boldsymbol{r}_{ik}\times\boldsymbol{p}_i\cdot\boldsymbol{\sigma}_i-\boldsymbol{r}_{ik}\times\boldsymbol{p}_k\cdot\boldsymbol{\sigma}_k)\frac{1}{r_{ik}}\frac{\mathrm{d}}{\mathrm{d}r_{ik}}\left\{\left(\sum_{\mathrm{S}} V_{ik}^{\mathrm{S}}+\sum_{\mathrm{V}} V_{ik}^{\mathrm{V}}\right)\right.$$
$$\left.\left.+\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_k\left(\sum_{(\mathrm{S}\tau)} V_{ik}^{\mathrm{S}\tau}+\sum_{(\mathrm{V}\tau)} V_{ik}^{\mathrm{V}\tau}\right)\right\}+\delta V_{ik}^{\mathrm{rel}}\right]\psi=\epsilon\psi \tag{21}$$

where $\delta V_{ik}^{\mathrm{rel}}$ contains all relativistic corrections which do not possess the form of the spin–orbit forces. It should be emphasized that the relative signs of V_{ik}^{S} and V_{ik}^{V} in the central (conservative) interaction and in the spin–orbit interaction are different: this fact appears to be very essential (see §§ 3 and 4). Pseudoscalar mesons make no contribution to the main terms of equation (21): they contribute only to $\delta V_{ik}^{\mathrm{rel}}$.

In the case of the vector–isovector (eg ρ meson) the additional contribution of the tensor coupling to the spin–orbit interaction should be taken into account. Using equation (4) the sum of the operators (19) and (20) for the vector–isovector meson may be written in the following form:

$$-\frac{\hbar}{8m^2c^2}\frac{1}{r_{ik}}\frac{\mathrm{d}V_{ik}^{\mathrm{V}\tau}}{\mathrm{d}r_{ik}}(\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_k)\{3\boldsymbol{r}_{ik}\times\boldsymbol{p}_{ik}(\boldsymbol{\sigma}_i+\boldsymbol{\sigma}_k)-\boldsymbol{r}_{ik}\times(\boldsymbol{p}_i+\boldsymbol{p}_k)\cdot(\boldsymbol{\sigma}_i-\boldsymbol{\sigma}_k)\}. \tag{22}$$

The tensor coupling will be taken into account, if the factor 3 in equation (22) is replaced by $3+4f^{\mathrm{V}\tau}/g^{\mathrm{V}\tau}$, $f^{\mathrm{V}\tau}$ and $g^{\mathrm{V}\tau}$ being, respectively, the tensor and vector coupling constants for the vector–isovector meson (Ueda and Green 1968, Ingber 1968).

If we realize the same transformations which were carried out in deriving equation (17), we shall obtain the single-particle spin–orbit potential in the following form:

$$V_{\mathrm{so}}^{(0)}(i)=\frac{\pi}{3}\frac{\hbar^2}{m^2c^2}\frac{1}{r}\frac{\mathrm{d}\rho^{(0)}}{\mathrm{d}r}\int_0^\infty\frac{\mathrm{d}}{\mathrm{d}x}\left[\left(\sum_{\mathrm{S}} V_{ik}^{\mathrm{S}}(x)+\sum_{\mathrm{V}} V_{ik}^{\mathrm{V}}(x)\right)\right.$$
$$\left.-\tau_3^i\frac{N-Z}{A}\left\{\sum_{(\mathrm{S}\tau)} V_{ik}^{\mathrm{S}\tau}(x)+\sum_{(\mathrm{V}\tau)}\left(1+2\frac{f^{\mathrm{V}\tau}}{g^{\mathrm{V}\tau}}\right)V_{ik}^{\mathrm{V}\tau}(x)\right\}\right]x^3\,\mathrm{d}x\boldsymbol{l}\cdot\boldsymbol{\sigma}, \tag{23}$$

where $A-1=N+Z$; τ_3^i is the projection of the isospin of the ith nucleon ($\tau_3^i/2=\pm\frac{1}{2}$, where the positive sign corresponds to the proton).

Thus, we obtained the single-particle spin–orbit operator of the Thomas form (equations (17) and (23)) with the gradient of potential replaced by the gradient of the matter density just as in the case of phenomenological introduction of the spin–orbit forces (Blin-Stoyle 1955). However, in contrast to the phenomenological approach (Blin-Stoyle 1955) we did not introduce the additional spin–orbit forces (with fitting parameters) into the nucleon–nucleon interaction. In formulae (17) and (23) V_{ik} or $V_{ik}^{(\mathrm{S},\mathrm{V})}$ are the conservative nucleon–nucleon interactions, that is, formulae (17) and (23) are rather 'rigid': all magnitudes in these formulae can be obtained from the data independent of the spin–orbit interaction, and the agreement (or absence of agreement) with experiment may serve as a verification of our initial assumption concerning the relativistic origin of the spin–orbit interaction in nuclei.

3. Average field and spin–orbit interaction

The definition of the average field by equation (7) is actually symbolic. Now we shall specify it.

Using the wavefunction in the Hartree approximation

$$\Psi(\xi_1, \xi_2, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots, \xi_A) = \prod_{k(k\neq i)} \psi_k(\xi_k) \tag{24}$$

(where ξ stands for space, spin and isospin variables), we shall define the average field $V_{\mathrm{av}}(i)$ as the expectation value of the operator $\Sigma_{k(k\neq i)} V_{ik}$ (see also Krutov 1972):

$$\begin{aligned}\sum_{k(k\neq i)} V_{ik} = V_{\mathrm{av}}(i) + V_{\mathrm{res}}(i) &= \sum_{k\leqslant A(k\neq i)} \int \mathrm{d}\xi' \psi_k^*(\xi') V_{ik}(\xi, \xi') \psi_k(\xi') + V_{\mathrm{res}}(i) \\ &= \left\langle \Psi^{(n)} \middle| \sum_{k(k\neq i)} V_{ik} \middle| \Psi^{(n)} \right\rangle + V_{\mathrm{res}}^{(n)}(i),\end{aligned} \tag{25}$$

where the superscripts $n = 0, 1, 2, \ldots$ stand for the order of approximation of the self-consistency procedure.

We shall utilize here, as well as in the calculations of the spin–orbit interaction, the short range approximation. Then, limiting ourselves in the expansion of the type (16) to the first term and using either the scalar interaction V_{ik} or replacing, in accordance with equation (21), V_{ik} by OBE potentials we shall obtain:

$$V_{\mathrm{av}}(r) \simeq \begin{cases} 4\pi\rho^{(0)}(r) \displaystyle\int_0^\infty V_{ik}(x) x^2 \,\mathrm{d}x & (26a) \\ 4\pi\rho^{(0)}(r) \displaystyle\int_0^\infty \left\{ \left(\sum_{\mathrm{S}} V_{ik}^{\mathrm{S}}(x) - \sum_{\mathrm{V}} V_{ik}^{\mathrm{V}}(x) \right) \right. & \\ \left. - \tau_3^i \frac{N-Z}{A} \left(\sum_{(\mathrm{S}\tau)} V_{ik}^{(\mathrm{S}\tau)}(x) - \sum_{(\mathrm{V}\tau)} V_{ik}^{(\mathrm{V}\tau)}(x) \right) \right\} x^2 \,\mathrm{d}x. & (26b) \end{cases}$$

It should be noted that calculating the average field we limit ourselves to the static limits of the OBE interactions, the corrections which are not taken into account are either small or are equal to zero (for the nuclei considered) in the Hartree approximation.

Now we shall compare equations (17) and (26*a*) on the one hand and equations (23) and (26*b*) on the other. These pairs of equations differ essentially from each other. In the case of equations (17) and (26*a*) we come to the result of the Thomas coupling. Actually, performing integration by parts in equation (17)

$$\int_0^\infty \frac{\mathrm{d}V_{ik}}{\mathrm{d}x} x^3 \,\mathrm{d}x = -3 \int_0^\infty V_{ik}(x) x^2 \,\mathrm{d}x$$

and taking into account equation (26*a*), we obtain from equation (17)

$$V_{\mathrm{so}}^{(0)}(i) \simeq -\frac{\hbar^2}{4m^2c^2} \frac{1}{r} \frac{\mathrm{d}V_{\mathrm{av}}}{\mathrm{d}r} \boldsymbol{l} \cdot \boldsymbol{\sigma}. \tag{27}$$

Thus, in the case of equations (17) and (26*a*) it is impossible to derive simultaneously (ie, to obtain from the same two-particle forces V_{ik}) the values of $V_{\mathrm{so}}(i)$ and $V_{\mathrm{av}}(i)$ which are in reasonable agreement with the experiment.

At the same time such derivation is possible in the case of equations (23) and (26*b*). Indeed, in accordance with equations (23) and (26*b*), the contributions of vector and

scalar mesons to the spin–orbit interaction have the same signs and their contributions to the average field have opposite signs. Thus, the situation may occur when these mesons compensate each other (to a considerable extent) contributing to the average field while their contributions to the spin–orbit potential are summed up.

Now we shall discuss the possibility of taking into account the exchange forces in calculations of $V_{av}(i)$. Let the two-particle interaction have the form of the mixture of the forces of Wigner, Bartlett, Heisenberg and Majorana:

$$V_{ik} = v(r_{ik})\left(G_W + G_B\frac{1+\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_k}{2} - G_H\frac{1+\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_k}{2} - G_M\frac{(1+\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_k)(1+\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_k)}{4}\right), \tag{28}$$

where $G_{W,B,H,M}$ are the strengths of the components normalized by the usual condition:

$$G_W + G_B + G_H + G_M = 1.$$

Then from equation (25) we have in the Hartree approximation

$$V_{av}(i) = \sum_{k\leqslant A(k\neq i)} (G_W + G_B\delta_{\sigma_i,\sigma_k} - G_H\delta_{\tau_i,\tau_k} - G_M\delta_{\sigma_i,\sigma_k}\delta_{\tau_i,\tau_k}) \int \psi_k^*(\boldsymbol{r}_k)v(r_{ik})\psi_k(\boldsymbol{r}_k)\,d\boldsymbol{r}_k, \tag{29}$$

the sense of the Kronecker symbols being evident. Assuming, as we have done previously, that the interaction is a short range one, instead of equation (26*a*) we shall obtain:

$$V_{av}(\boldsymbol{r}) \simeq 4\pi\rho^{(0)}(\boldsymbol{r})\int_0^\infty v(x)x^2\,dx\left(G_W + \frac{1}{2}G_B - G_H\frac{N}{A} - \frac{1}{2}G_M\frac{N}{A}\right), \tag{30}$$

where the ith nucleon is considered to be a neutron (for a proton N should be replaced by Z).

As for consideration of the exchange forces in equation (17), we shall use (in § 4, calculating $V_{so}(i)$ in accordance with equation (17)) the interaction (28), taking into account the transformation from equation (17) to (27). However, we shall attach only illustrative importance to this consideration of the exchange forces in the framework of equation (17). Indeed, although equation (28) is the most general form of notation of the conservative central forces, we cannot use equation (28) in equation (17) directly, since equation (17) is obtained from the semirelativistic equation (2), where the exchange character of V_{ik} is not assumed. The proper consideration of the exchange components of the nuclear forces in calculations of $V_{so}(i)$ can be carried out only when the exchange forces are taken into account in the initial semirelativistic equation. It was just this procedure that was realized in equation (18) and that is why in the notation of $V_{so}(i)$, in accordance with equation (23), the exchange character of the nuclear forces is properly taken into account.

4. Calculations and discussion of the spin–orbit splittings and average field for near magic nuclei

Here we shall calculate the spin–orbit splittings of different states in the near magic nuclei ^{41}Ca and ^{209}Pb, as well as the values of the average field for the same nuclei.

According to equation (17) and equation (23), the doublet splitting of the single-particle level $|n, l = j \mp \frac{1}{2}\rangle$ for the nuclei with one nucleon above the closed shells is equal to

$$\Delta E_{nl} = -(2l+1)\left(\int_0^\infty |f_{nl}(r)|^2 \frac{\mathrm{d}\rho^{(0)}}{\mathrm{d}r} r\,\mathrm{d}r\right)\left(\frac{\pi\hbar^2}{3m^2c^2}\int_0^\infty \frac{\mathrm{d}\overline{V}_{ik}(x)}{\mathrm{d}x} x^3\,\mathrm{d}x\right), \tag{31}$$

where $\overline{V}_{ik}(x) = V_{ik}(x)$ in the case of equation (17), and

$$\overline{V}_{ik}(x) = \left(\sum_{\mathrm{S}} V_{ik}^{\mathrm{S}}(x) + \sum_{\mathrm{V}} V_{ik}^{\mathrm{V}}(x)\right) - \tau_3^i \frac{N-Z}{A}\left(\sum_{(\mathrm{S}\tau)} V_{ik}^{\mathrm{S}\tau}(x) + \sum_{(\mathrm{V}\tau)} \left(1 + 2\frac{f^{\mathrm{V}\tau}}{g^{\mathrm{V}\tau}}\right) V_{ik}^{\mathrm{V}\tau}(x)\right)$$

in the case of equation (23), the function $f_{nl}(r)$ is the radial wavefunction of an odd nucleon in the field $V_{\mathrm{av}}(r)$.

At first we shall discuss the calculation of the first integral in equation (31). Similarly to Blin-Stoyle (1955) we utilize the following model: an infinite well of radius R is taken for obtaining the functions $f_{nl}(r)$, and the nuclear density is taken in the trapezoidal form (close to the realistic Fermi distribution):

$$\rho^{(0)}(r) \simeq \begin{cases} \rho_0 & \text{for} \quad 0 \leqslant r \leqslant R(1-\tau) \\ \rho_0 \dfrac{R-r}{R\tau} & \text{for} \quad R(1-\tau) \leqslant r \leqslant R, \end{cases} \tag{32}$$

where τ is the relative thickness of the surface layer in which the density decreases from ρ_0 to zero. We assume that $R = 1{\cdot}33A^{1/3}$ fm (Blin-Stoyle takes the nuclear radius equal to $1{\cdot}45A^{1/3}$ fm), since this value of R for $\tau = 0{\cdot}4$ corresponds to the half density radius $R_{1/2} = 1{\cdot}07A^{1/3}$ fm. The latter value is close to the experimental values of $R_{1/2}$ of the charge distribution in the nucleus (Hofstadter 1956).

To estimate the accuracy of the approximations made in the derivation of $f_{nl}(r)$ and $\rho^{(0)}(r)$, we carried out calculations of the 1i level splitting in ^{209}Pb using the Woods–Saxon potential and the Fermi distribution of density. The wavefunctions for the $1i_{11/2}$ and $1i_{13/2}$ neutron states were taken from the paper by Blomqvist and Wahlborn (1960), and the Fermi density distribution parameters

$$\rho^{(0)}(r) = \rho_0\left\{1 + \exp\left(\frac{r-R_{1/2}}{a}\right)\right\}^{-1}, \tag{33}$$

$\rho_0 = 0{\cdot}17$ fm^{-3}, $a = 0{\cdot}54$ fm, $R_{1/2} = 6{\cdot}5$ fm, were taken from the monograph by Bohr and Mottelson (1969). It was found that for the 1i level in ^{209}Pb the values ΔE_{nl} calculated with the Fermi distribution and with the wavefunctions of the Woods–Saxon potential differ only by 12% from the values ΔE_{nl} calculated in our simplified model (with $\tau = 0{\cdot}4$). Therefore we consider the model which uses the rectangular well and the trapezoidal density distribution to be a satisfactory model for practical calculations (since ambiguity in the choice of the two-particle interaction leads at present to a considerably larger uncertainty in the final result than 12%) and we shall utilize it in what follows.

Now we shall proceed to calculations of ΔE_{nl} and V_{av}. At first we shall consider in equation (31) the case with $\overline{V}_{ik}(x) = V_{ik}(x)$, and in calculations of the average field we shall use equation (30) as a refined version of equation (26*a*). We shall utilize the following two variants of the conservative forces.

(i) The Volkov (1965) forces which were used in calculations of the ground state energies of the 1p shell nuclei:

$$V_{ik}(\xi) = \{1 - G_M - G_M \tfrac{1}{4}(1 + \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_k)(1 + \boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_k)\} \left\{ -v_1 \exp\left(-\frac{x^2}{B_1^2}\right) + v_2 \exp\left(-\frac{x^2}{B_2^2}\right) \right\} \tag{34}$$

where $G_M = 0{\cdot}6$, $v_1 = 60{\cdot}6$ MeV, $v_2 = 61{\cdot}1$ MeV, $B_1 = 1{\cdot}8$ fm and $B_2 = 1{\cdot}01$ fm.

(ii) The Barker (1966) exchange mixture used in intermediate coupling shell model calculations; these forces have the form of equation (28) with the following values of the parameters:

$$\begin{aligned} &v(x) = -v_0 \exp\{-(x/B)^2\}, \qquad v_0 = 70{\cdot}8 \text{ MeV}, \qquad B = 1{\cdot}48 \text{ fm}, \\ &G_W = 0{\cdot}075, \qquad G_B = 0{\cdot}5, \qquad G_H = -0{\cdot}325, \qquad G_M = 0{\cdot}75. \end{aligned} \tag{35}$$

In table 1 we present the values of the spin–orbit splittings ΔE_{nl} obtained with the Volkov and Barker forces for a number of states of ^{41}Ca and ^{209}Pb (we assume that $\tau = 0{\cdot}4$). The depths of the average field $\tilde{V}_{av}$ (within the interval $0 \leqslant r \leqslant R(1-\tau)$) obtained for the same nuclei, in accordance with equation (30), are presented in the table as well.

Table 1. The neutron spin–orbit splittings and average field values in the case when the Volkov and Barker forces are used as two-particle interactions†

Form of the two-particle interaction	Volkov forces	Barker forces	Experimental data
$\Delta E_{1f}(^{41}\text{Ca})$	0·621	0·597	6·50
$\Delta E_{2p}(^{41}\text{Ca})$	0·148	0·142	2·00
$\Delta E_{2g}(^{209}\text{Pb})$	0·129	0·139	2·47
$\Delta E_{1i}(^{209}\text{Pb})$	0·416	0·446	4·59
$\Delta E_{3d}(^{209}\text{Pb})$	0·0731	0·0782	0·98
$\tilde{V}_{av}(^{41}\text{Ca})$	−73	−70	—
$\tilde{V}_{av}(^{209}\text{Pb})$	−64	−68·7	—
$V_{av}^{eff}(^{41}\text{Ca})$	−58·1	−55·7	−46·1
$V_{av}^{eff}(^{209}\text{Pb})$	−50·9	−54·7	−53·4

† All values are given in MeV. The experimental values of ΔE_{nl} for ^{41}Ca are taken from Pearson *et al* (1969), those for ^{209}Pb from Gillet *et al* (1967). The experimental values of V_{av}^{eff} for ^{40}Ca and ^{208}Pb are taken from the optical model analysis of the 30 MeV proton scattering (Greenlees and Pyle 1966).

According to the approximate equations (26) and (30), $V_{av}(r)$ reproduces the form of $\rho(r)$; for this reason the average field obtained here differs slightly in width from the conventional empirical potentials. This difference, certainly, will manifest itself in the comparison of the depths of the potentials. To perform the proper comparison of our results with empirical data, we utilize the following relation:

$$\tilde{V}_{av}(R_{1/2})^2 = V_{av}^{eff}(R_{1/2}^{eff})^2, \tag{36}$$

and (in table 1 and in other tables) we give the values V_{av}^{eff} as well, considering $R_{1/2}^{eff} = 1{\cdot}2A^{1/3}$ fm (the same or close half-value radii of the central potential are used in the optical model analysis, see, for example, Greenlees and Pyle (1966)).

As is seen from table 1 the values of the spin–orbit splittings obtained with the Volkov and Barker forces are smaller by one order of magnitude than the experimental splittings whereas the calculated average fields prove to be close to the empirical fields. We have already discussed this discrepancy in § 3.

Now we shall proceed to consideration of the one-boson-exchange potentials of meson theory as a two-particle interaction. In practical calculations we have utilized two PSV variants of OBEP, which we shall discuss separately.

At first we shall consider the calculations performed with the PSV variant of OBEP which was applied by Green and Sawada (1967) to the description of the nucleon–nucleon scattering data within the energy region 0–330 MeV. Green and Sawada denote this variant as a one-parameter model, in what follows we shall call this interaction the forces of Green and Sawada. These forces achieve good fits to scattering data, they reproduce well the S wave shifts and all higher partial wave shifts with the exception of the $^1P_1$ and, to a lesser extent, the $^1D_2$ wave (Green and Sawada 1967). In this case the functions V_{ik}^{V} and V_{ik}^{S} have the form:

$$V_{ik}^{(V,S)} = -\frac{g^2}{r_{ik}}\left(\exp(-\mu r_{ik}) - \frac{U^2-\mu^2}{U^2-\Lambda^2}\exp(-\Lambda r_{ik}) + \frac{\Lambda^2-\mu^2}{U^2-\Lambda^2}\exp(-U r_{ik})\right) \tag{37a}$$

with the following values of the parameters:

$$\mu_\omega = 783 \text{ MeV}, \qquad \mu_S = 600 \text{ MeV}, \qquad \Lambda_\omega = \Lambda_S = 2m = 1880 \text{ MeV},$$
$$U_\omega = U_S = 20m, \qquad g_S^2 = g_\pi^2 = 14{\cdot}7, \qquad g_\omega^2 = g_S^2(\mu_\omega/\mu_S)^2 = 25. \tag{37b}$$

In table 2 we present the values of the spin–orbit splittings and average fields obtained with the two-particle forces of Green and Sawada (the calculations are carried out for three values of τ).

Table 2. The spin–orbit splittings and average field values for the case when the forces of Green and Sawada are used as two-particle interactions†

Thickness of the surface layer (τ)	0·3	0·4	0·5	Experimental data
$\Delta E_{1f}(^{41}\text{Ca})$	6·30	9·32	11·1	6·50
$\Delta E_{2p}(^{41}\text{Ca})$	2·18	2·22	2·20	2·00
$\Delta E_{2g}(^{209}\text{Pb})$	1·97	2·22	3·86	2·47
$\Delta E_{1i}(^{209}\text{Pb})$	5·91	7·08	7·12	4·59
$\Delta E_{3d}(^{209}\text{Pb})$	0·842	1·28	1·67	0·98
$\tilde{V}_{av}(^{41}\text{Ca})$	−39·2	−45·3	−52·7	—
$\tilde{V}_{av}(^{209}\text{Pb})$	−39·2	−45·3	−52·7	—
$V_{av}^{eff}(^{41}\text{Ca})$	−34·8	−36·1	−36·6	−46·1
$V_{av}^{eff}(^{209}\text{Pb})$	−34·8	−36·1	−36·6	−53·4

† All values are given in MeV.

We have also utilized as two-particle forces the PSV variant of OBEP which was considered by Ueda and Green (1968) (here we use their model III, the results for the first two models being close to those for III). These forces (we shall denote them as the forces of Ueda and Green) were applied by Ueda and Green to the description of the p–p and

n–n scattering data and the deuteron data (deuteron binding energy, quadrupole moment, magnetic moment etc), most of these data are reproduced almost quantitatively. The results of our calculations of ΔE_{nl} and V_{av} obtained with the forces of Green and Ueda are given in table 3.

Table 3. The spin–orbit splittings and average field values for the case when the forces of Ueda and Green are used as two-particle interactions†

τ	0·3	0·4	0·5	Experimental data
$\Delta E_{1f}(^{41}\text{Ca})$	1·64	2·42	2·90	6·50
$\Delta E_{2p}(^{41}\text{Ca})$	0·569	0·577	0·574	2·00
$\Delta E_{2g}(^{209}\text{Pb})$	0·526	0·588	1·02	2·47
$\Delta E_{1i}(^{209}\text{Pb})$	1·56	1·87	1·89	4·59
$\Delta E_{3d}(^{209}\text{Pb})$	0·224	0·339	0·443	0·98
$\tilde{V}_{av}(^{41}\text{Ca})$	−73·8	−85·4	−99·2	—
$\tilde{V}_{av}(^{209}\text{Pb})$	−74·5	−86·1	−100	—
$V_{av}^{eff}(^{41}\text{Ca})$	−65·5	−68	−69	−46·1
$V_{av}^{eff}(^{209}\text{Pb})$	−66·1	−68·6	−69·5	−53·4

† All values are given in MeV.

As is seen from tables 2 and 3, rather satisfactory values both of the spin–orbit potential and the average field may be obtained in the case when the PVS models of OBEP are used. Certainly, all PVS models should be considered for the present as preliminary ones since it is not quite clear which mesons and resonances should be included in OBEP. Therefore for the present our calculations do not allow us to make the indisputable conclusion that the nuclear single-particle spin–orbit potential is of relativistic origin. But nevertheless our results may be considered as a telling argument in favour of such an origin.

In conclusion we shall add the following. In a number of papers of one of the authors (see eg Krutov 1968, 1972, Krutov and Zackrevsky 1969) the semiphenomenological model of the nucleus was developed and applied successfully to the description of a wide range of manifold experimental data with a small number of adjustable parameters. However, in this model the basis Hamiltonian consisting of the single-particle potential and residual interactions is given in a parametric form, and not calculated. In the present paper the first step taken is to calculate this basis Hamiltonian, namely, starting from the nucleon–nucleon interactions we have calculated with reasonable success the average field and spin–orbit potential, that is, the main part of the single-particle potential (in the Hartree approximation so far). The authors believe that in a following paper they will be able to present calculations of the average field and spin–orbit potential in the Hartree–Fock approximation, to calculate the isobaric spin potential and to consider the isospin dependence of the spin–orbit potential as well.

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